Numerical Differentiation

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if h is small

$$f'(x) \simeq \frac{f(x+h) - f(x)}{h}$$

Taylor series

If $f \in C^{N+1}[a,b]$ and $x \in [a,b]$
$x_0 \in [a,b]$

Then The Taylor series can be written in form:-

$$f(x)\Big|_{x=x_0} = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + E_n$$

Truncation error

for $(x+h)$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \cdots$$

(1) Subtract $f(x-h)$ from $f(x+h)$, we get

$$f(x+h) - f(x-h) = 2h f'(x) + \frac{h^3}{3!} f'''(x)$$

$$\boxed{f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O h^2}$$

(2) Add $f(x-h)$ to $f(x+h)$

$$f(x+h) + f(x-h) = 2f(x) + \frac{2h^2}{2!} f''(x)$$

$$\boxed{f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O h^2}$$

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if we want to increase the approximation accuracy we can use the values of ~~the~~ functions at

$(x+2h)$, $(x+h)$, $(x-h)$ & $(x-2h)$

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{2h^3}{3!} f'''(x) + O h^5$$

$$f(x+2h) - f(x-2h) = 2hf'(x) + \frac{16h^3}{3!} f'''(x) - O h^5$$

$$\boxed{f'(x) = \frac{f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)}{12h} + O h^5}$$

Also :-

$$\boxed{f''(x) = \frac{1}{12h^2}\left(-f(x+2h) + 16f(x+h) - 30f(x) + 16f(x-h) - f(x-2h)\right) + O h^4}$$

Forward and Backward Difference.

for The three points x_0, x_1 & x_2 to Compute The derivative $f'(x_0)$ using lagrange interpolation

The interpolation Takes The form

$$f(x) = f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

and the differentiate

$$f'(x) = f(x_0)\frac{(x-x_1)+(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$+ f(x_1)\frac{(x-x_0)+(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$+ f(x_2)\frac{(x-x_0)+(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

Put $x=x_0$ & $x_1 = x_0+h$, $x_2 = x_1+h = x_0+2h$

$$f'(x_0) = f(x_0)\frac{-3h}{2h^2} + f(x_0+h)\frac{-2h}{-h^2}$$

$$+ f(x_0+2h)\frac{-h}{h^2}$$

$$\boxed{f'(x_0) = \frac{-3f(x_0)+4f(x_0+h)-f(x_0+2h)}{2h}}$$

Note this approximation with an accuracy of $O(h^2)$.

Also, we can use the same approach to find $f'(x)$ using points to the left of x_0. Just put $(h=-h)$

$$\boxed{f'(x_0) = \frac{3f(x_0)-4f(x_0-h)+f(x_0-2h)}{2h}}$$

Also we can find higher derivatives by differentiating the interpolating Polynomial more than once:-

$$\boxed{f''(x_0) = \frac{2f(x) - 5f(x+h) + 4f(x+2h) - f(x+3h)}{h^2}}$$

Example:-

using the following data to approximate

$f'(2), f''(2), f'(1.8), f'(2.2)$

$f''(2.2)$

	x_0-2h	x_0-h	x_0	x_0+h	x_0+2h
x	1.8	1.9	2	2.1	2.2
y	10.8893	12.7032	14.7781	17.149	19.8550

Sol

to get $f'(2)$ & $f''(2)$

$$f'(x) = \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} + O h$$

OR backward.

$$f'(x) = \frac{3f(x) - 4f(x-h) + f(x-2h)}{2h} + Oh$$

OR Central

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + Oh^2$$

OR using 5 points.

$$f'(x) = \frac{f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)}{12h} + Oh^4$$

Note The exact previous results for $f(x) = xe^x$